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LETTER TO THE EDITOR

Uncertainty relations for wavefunctions

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Abstract. It is proved that when the Hilbert space of quantum mechanics is a reproducing kernel Hilbert space of functions defined on the phase space, the generating function of the entropy can be used as a measure of the concentration of the wavefunctions in the phase space. The sharp upper bound obtained for the generating function of the entropy proves that the wavefunctions cannot be arbitrarily peaked in the phase space. The most peaked wavefunctions are those defined by the reproducing kernels.

There are many cases in which quantum theory is formulated on a subspace of the Hilbert space of square-integrable functions defined on the phase space [1]. This subspace is determined by a projection operator which is an integral operator. The kernel of this operator is called the reproducing kernel since by definition it reproduces all elements of the subspace. The best known example is the Bargmann representation of the canonical commutation relations [2] which is a holomorphic coherent-state representation associated with the square-integrable representations of the Heisenberg–Weyl group [3]. Other holomorphic coherent-state representations often considered in quantum mechanics are those associated with the square-integrable representations of the groups SU(1, 1) and SU(2). In these examples the phase space Δ is $\mathbb{C}$, the unit disc and the Riemann sphere $\mathbb{C} \cup \{\infty\}$ respectively. The wavefunctions are the elements of the reproducing kernel Hilbert space $\tilde{\mathscr{H}}_\beta$ of functions defined on Δ which are of the following form:

$$\psi_\beta(x, y) = f(z)k(|z|^2)^{-1/2} \tag{1}$$

with $z = x + \mathrm{i}y$ and f and k holomorphic functions on Δ. These wavefunctions are square-integrable with respect to the Lebesgue measure $\mathrm{d}x\,\mathrm{d}y$ on Δ. The parameter β defines uniquely the generalised dimension $\dim(\beta)$ of the corresponding square-integrable representation of each of the three groups enumerated above [3]. Namely, $\dim(\beta) = \beta$ for the Heisenberg–Weyl group and $\dim(\beta) = \beta - 1$ with β a real positive number for the group SU(1, 1) and for the group SU(2) the generalised dimension coincides with the ordinary dimension: $\dim(\beta) = 2\beta + 1$, where β takes integer and semi-integer values.

The uncertainty relations for wavefunctions on the phase space are a quantitative formulation of the intuitive idea that the probability density defined on the phase space by the square of the modulus of such a wavefunction cannot be arbitrarily peaked in the phase space.

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In the present letter we shall use as a measure of the concentration of the probability density on Δ, defined by $P_\beta(x, y) = |\psi_\beta(x, y)|^2$, the generating function of the entropy [4]:

$$G_r(P_\beta) = \pi^{-1} \dim(\beta) \int_\Delta \int P(x, y)^r \, dx \, dy \tag{2}$$

where r is a real positive number.

The uncertainty relations are sharp inequalities of the following type:

$$G_r(P_\beta) \leqslant F(r, \beta) \tag{3}$$

with $F(r, \beta)$ independent of P_β and such that there exist wavefunctions for which the equality holds in (3).

Let us denote by $\mathscr{H}_\beta$ the Hilbert space of holomorphic functions f on Δ which appear in (1) and with the norm defined by

$$\|f\|_\beta^2 = \pi^{-1} \dim(\beta) \int_\Delta \int |\psi_\beta(x, y)|^2 \, dx \, dy. \tag{4}$$

The Hilbert space $\mathscr{H}_\beta$ is a reproducing kernel Hilbert space with the reproducing kernel given by $k_\beta(\bar{w}z)$ with $w \in \Delta$.

In order to obtain the uncertainty relations of the kind described above we shall use the following theorem, proved in [5].

If $f \in \mathscr{H}_\beta$ and $h \in \mathscr{H}_{\beta'}$ then $fh \in \mathscr{H}_{\beta+\beta'}$ and

$$\|fh\|_{\beta+\beta'} \leqslant \|f\|_\beta \|h\|_{\beta'}. \tag{5}$$

The equality holds if and only if either $fh = 0$ or f and h are of the forms $f = C_1 k_\beta(\bar{w}z)$, $h = C_2 k_\beta(\bar{w}z)$ for some $w \in \Delta$ and some non-zero constants C_1 and C_2.

As a corollary one obtains [5] for any natural number n:

$$\|f^n\|_{n\beta} \leqslant \|f\|_\beta^n \tag{6}$$

with the equality either for $f = 0$ or when $f = Ck_\beta(\bar{w}z)$ for some $w \in \Delta$ and some non-zero constant C.

In terms of the generating function for the entropy, the inequality (6) gives the following uncertainty relation:

$$G_n(P_\beta) \leqslant \dim(\beta)/\dim(n\beta). \tag{7}$$

The equality holds in (7) either when $n = 1$ or when

$$P_\beta(x, y) = |k_\beta(\bar{w}z)|^2 k_\beta(|z|^2)^{-1}. \tag{8}$$

Hence the most concentrated wavefunction is the reproducing kernel. This result is in agreement with intuition. Indeed, the reproducing kernel is a smooth analogue of a distribution with the probability density concentrated on a single point.

For the sake of completeness we give the functions $k_\beta(z)$ in each of the three cases considered above: $k_\beta(z) = \exp(\beta z)$ in the case of the Heisenberg-Weyl group, $k_\beta(z) = (1-z)^{-\beta}$ in the case of the SU(1, 1) group and $k_\beta(z) = (1+z)^{2\beta}$ in the case of the SU(2) group.

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